Fractional Derivatives by Fourier Decomposition

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This notebook investigates the properties of non-integer differential operators using Fourier analysis.

```
In[31]:= << Graphics`Legend`
```

We'll start out on the interval (-∞,∞). f[x] is a normalizable function. Let's pick it to be a Gaussian for the sake of being concrete.

In[32]:= **f[x_] := e^{-x^2}**

F[k] is its Fourier transform.

In[33]:= **F[k_] := Evaluate[$\int_{-\infty}^{\infty} f[x] * e^{i*k*x}\, dx$]**

h[,α,x] is the $(\frac{d}{dx})^{\alpha}$ derivative of f[x]. We'll start by assuming α is a real number greater than 0. (α<0 corresponds to integration.) We'll also restrict our attention to x∈Reals, $\alpha \geq 0$ for the sake of *Mathematica*.

In[34]:= **h[α_, x_] :=**
Evaluate[
$(-i)^{\alpha}$ *
Integrate[$\frac{(k)^{\alpha} * F[k] * e^{-i*k*x}}{2*\pi}$, {k, -∞, ∞}, Assumptions → {x ∈ Reals, α ∈ Reals, α ≥ 0}]
]

Look, Ma: hypergeometric functions.

In[35]:= **h[α, x]**

Out[35]= $\frac{1}{2\sqrt{\pi}}\left((-i)^{\alpha}\left(2\, i\, ((-2)^{\alpha}-2^{\alpha})\, x\, \text{Gamma}\left[1+\frac{\alpha}{2}\right] \text{Hypergeometric1F1}\left[1+\frac{\alpha}{2}, \frac{3}{2}, -x^2\right] + ((-2)^{\alpha}+2^{\alpha})\, \text{Gamma}\left[\frac{1+\alpha}{2}\right] \text{Hypergeometric1F1}\left[\frac{1+\alpha}{2}, \frac{1}{2}, -x^2\right]\right)\right)$

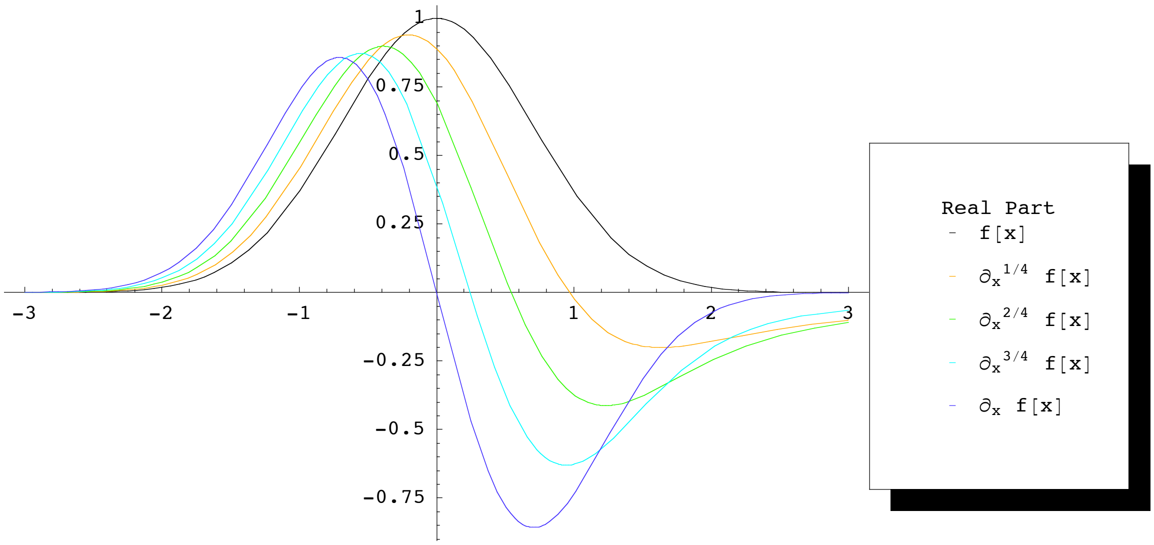
H[α,x] is h[α,x] explicitly defined to avoid convergence issues in *Mathematica*.

In[36]:= **H[α_, x_] :=** $\frac{1}{2\sqrt{\pi}}\left((-i)^{\alpha}\left(2\, i\, ((-2)^{\alpha}-2^{\alpha})\, x\, \text{Gamma}\left[1+\frac{\alpha}{2}\right] \text{Hypergeometric1F1}\left[1+\frac{\alpha}{2}, \frac{3}{2}, -x^2\right] + ((-2)^{\alpha}+2^{\alpha})\, \text{Gamma}\left[\frac{1+\alpha}{2}\right] \text{Hypergeometric1F1}\left[\frac{1+\alpha}{2}, \frac{1}{2}, -x^2\right]\right)\right)$

In[37]:= **H[α, x];**

We here plot the real part of these functions because *Mathematica* calculates them to finite precision, and, in doing so, incorrectly gives them tiny (but non-zero) imaginary parts.

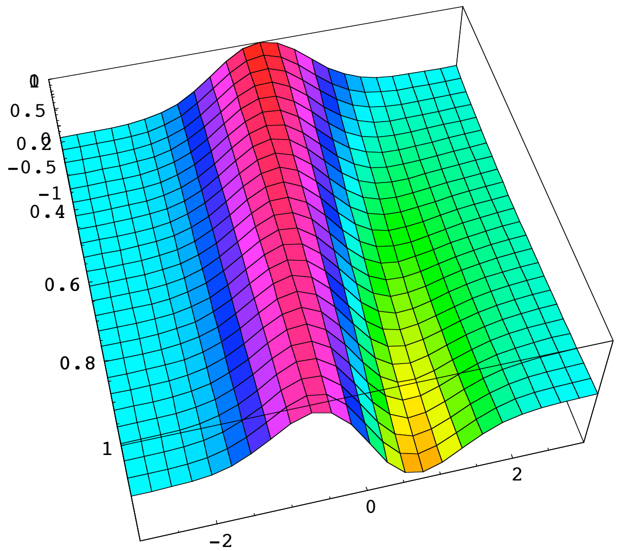
In[38]:=
```
Plot[
 {f[x], Re[H[1/4, x]], Re[H[1/2, x]], Re[H[3/4, x]], Re[H[1, x]]}, {x, -3, 3},
 PlotStyle → {GrayLevel[0], Hue[.1], Hue[.3], Hue[.5], Hue[.7]},
 PlotLegend → {"f[x]", "∂x^(1/4) f[x]", "∂x^(2/4) f[x]", "∂x^(3/4) f[x]", "∂x f[x]"},
 LegendPosition → {1, -.5},
 LegendBorderSpace → 10,
 LegendSpacing → 3,
 LegendTextSpace → 15,
 LegendLabel → "Real Part"

]
```



Out[38]= - Graphics -

Here we plot the fractional derivatives of f[x] as a function of α on α = (0,1).

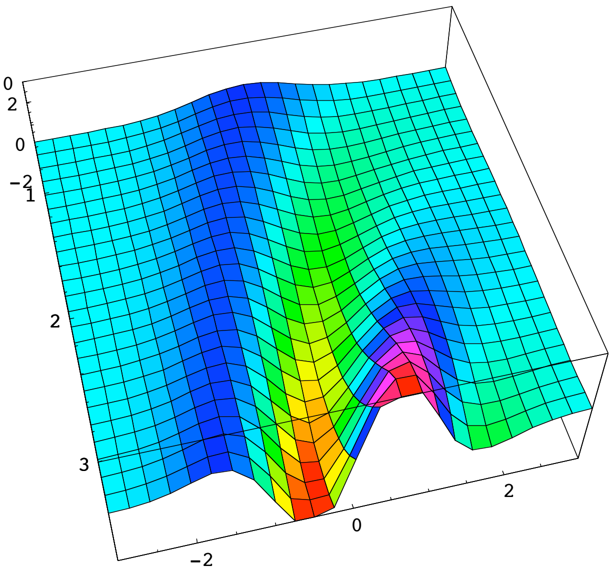
```
In[39]:= Plot3D[Re[H[α, x]],
          {α, 0, 1}, {x, -3, 3},
          Mesh → True,
          PlotRange → {-1, 1},
          ColorFunction → Hue,
          ViewPoint → {2, -.5, 2.7}
         ]
```




```
Out[39]= - SurfaceGraphics -
```

This is the same plot with the range extended to the interval $\alpha = (0,3)$.

```
In[40]:= Plot3D[Re[H[α, x]],
          {α, 0, 3}, {x, -3, 3},
          Mesh → True,
          PlotRange → {-3, 3},
          ColorFunction → Hue,
          ViewPoint → {2, -.5, 2.7}
         ]
```




```
Out[40]= - SurfaceGraphics -
```

From the same method of analysis, we can see that a Green's function for the differential operator $\hat{L} = (\partial_x)^\alpha$ is given by:

$$G_{(x-y)} = \int_{-\infty}^{\infty} \frac{dk}{2*\pi} \frac{1}{(i*k)^\alpha} e^{i*k*(x-y)}$$

Another interesting item to note in passing is that **a non-integer derivative of any polynomial is either 0 or not defined.**

Proof:
Consider the Fourier transform of x^n.

$$x^n = \int_{-\infty}^{\infty} \frac{dk}{2*\pi} * (-1)^n * \frac{e^{i*k*x}}{i^n} * (\partial_k)^n (\delta_{(k)})$$

And thus:

$$(\partial_x)^\alpha x^n = \int_{-\infty}^{\infty} \frac{dk}{2*\pi} * (-1)^n * (i*k)^\alpha * \frac{e^{i*k*x}}{i^n} * (\partial_k)^n (\delta_{(k)})$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2*\pi} * (-1)^{n-1} * \partial_k\left((i*k)^\alpha * \frac{e^{i*k*x}}{i^n}\right) * (\partial_k)^{n-1} (\delta_{(k)})$$

By integrating by parts, we can move the (integer) k derivatives from the right, (where they act on the k delta function,) to the left, where they act on $\left((i*k)^\alpha * \frac{e^{i*k*x}}{i^n}\right)$. Speaking somewhat loosely, the k derivative acting on this term will give you something of the form:

$$\text{(terms with k raised to some fractional power)}* * \frac{e^{i*k*x}}{i^n}$$

We can keep integrating by parts until the k delta function has no derivatives acting on it, and for all our labor, the integrand will still be of the form:

$$\text{(terms with k raised to some fractional power)}* * \frac{e^{i*k*x}}{i^n} * \delta_{(k)}$$

At this stage, we can just do the integral by removing the k delta function and setting to k=0 everywhere else in the integrand. If all the "terms with k raised to some fractional power" are positive powers of k, they will become 0 when we set k=0. If any of the "terms with k raised to some fractional power" are negative, then the expression becomes infinite, and so the fractional derivative does not exist. The last imaginable case is that maybe there is some term among the "terms with k raised to some fractional power" which is k raised to 0, (i.e. it is constant.) But upon reflection, we see that there will never be such a term. If α is **not** an integer, then no matter how many times we raise or lower the order of k by increments of one (by integrating by parts), we'll never get an integer power of k. Thus, we'll never get a term like k^0 as long as α is not an integer.

This is all interesting because we saw above that a Gaussian, which is expressable--by Taylor expansion--as an infinite sum of polynomials, has finite, non-vanishing fractional derivatives. But here we see that any polynomial by itself has only vanishing or non-existent fractional derivatives. If you imagine acting with a fractional derivative on each term in the Taylor expansion of the Gaussian function, you would find that you have an infinite number of infinite terms with oscillating sign. This is telling you that the Taylor expansion is not a practical way to take a fractional derivative of the Gaussian--i.e., the method is not well-defined. Rather, you have to work in the frequency domain in order to make sense of anything.

Keywords: Fourier analysis, frequency decomposition, fractional derivatives, non-integer derivatives, Plot3D